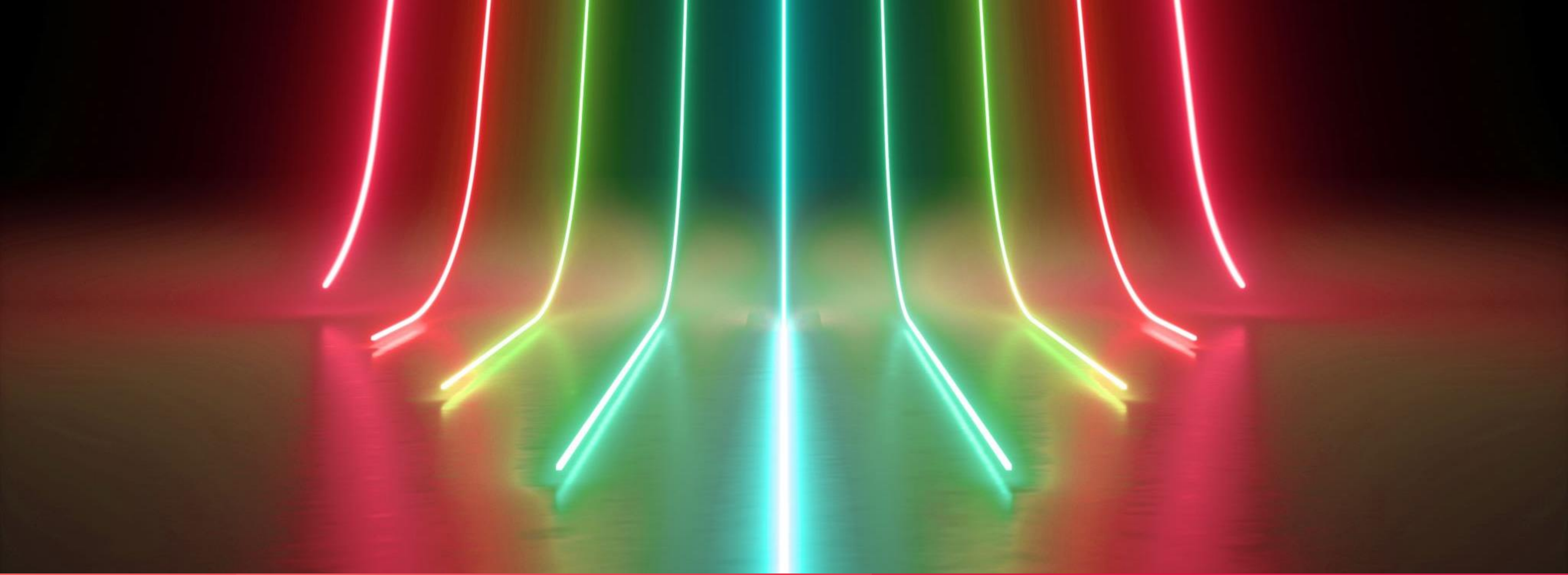

LYAPUNOV STABILITY THEORY: FUNDAMENTALS

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The Lyapunov System Stability

The Lyapunov stability is useful in nonlinear control systems and it expresses the condition for which a state system starts and remain close enough to the equilibrium

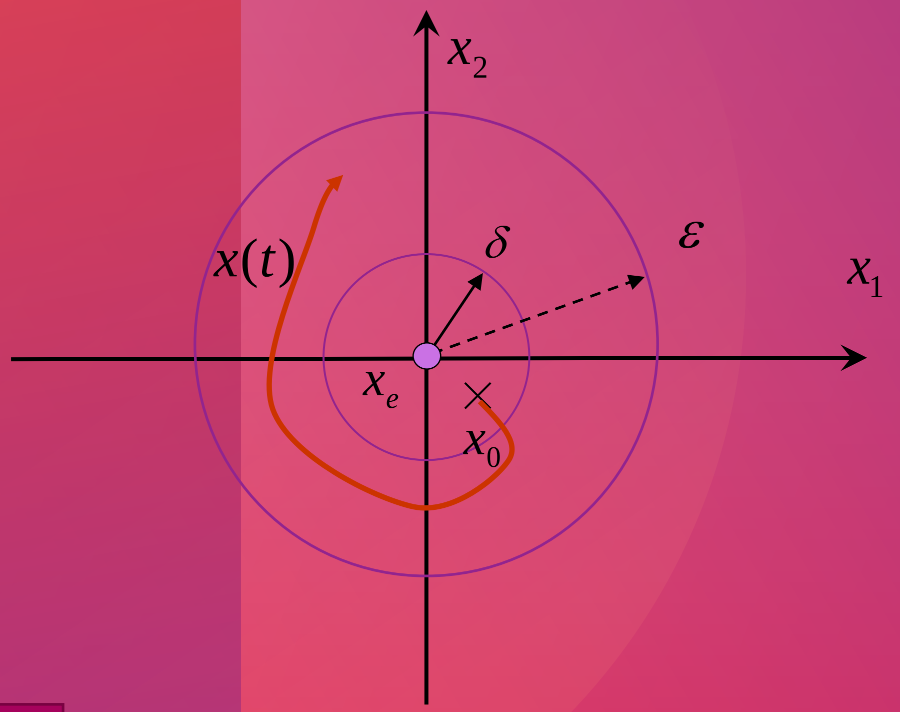
System state $x(t) \in X \subseteq R^n$

System function $\dot{x} = f(t), f: X \to R^n$

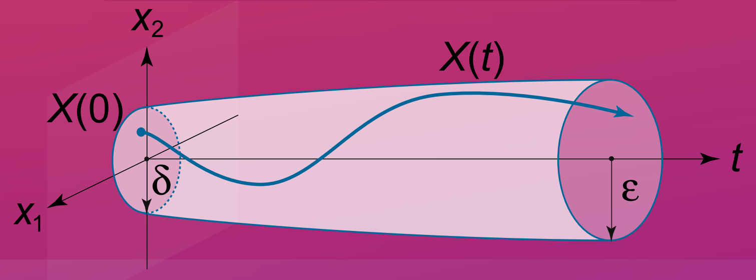
Equilibrium Condition

$$\forall \epsilon > 0, \exists \delta > 0,$$

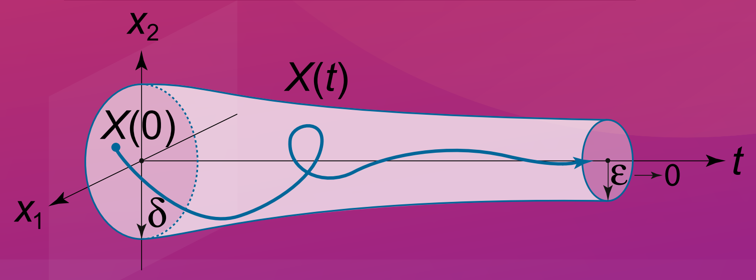
$$\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \epsilon, \forall t > 0$$



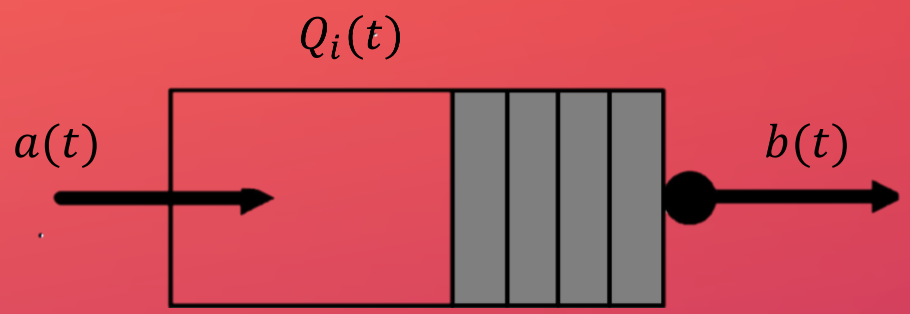
Lyapunov Stability



Asymptotic Stability



The Lyapunov in Wireless Queueing Systems



$V(x) = 0, \text{if } x = x_e$

$V(x) > 0, \text{if } x \neq x_e$

Lyapunov function

$$\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x)$$

$\dot{V}(x) \leq 0 \Rightarrow$ Lyapunov stability

$\dot{V}(x) < 0 \Rightarrow$ asymptotic stability

$Q(t+1) = \max[Q(t) - b(t), 0] + a(t)$ Queue dynamic

1. **Rate Stable:** A queue $Q_i(t)$ is rate stable if

$$\lim_{\tau \to \infty} \frac{Q_i(\tau)}{\tau} = 0$$ with probability 1

2. **Mean Rate Stable:** A queue $Q_i(t)$ is mean rate stable if

$$\lim_{\tau \to \infty} \frac{\mathrm{E}\{|Q_i(\tau)|\}}{\tau} = 0$$

3. **Strongly Stable:** A queue $Q_i(t)$ is strongly stable if

$$\lim_{t \to \infty} \sup \frac{1}{t} \sum_{\tau=1}^{t-1} \sum_{i=1}^{N} E\{|Q_i(\tau)|\} < \infty$$ Focuses on the expected time average

The Lyapunov in Wireless Queueing Systems

$$\overline{g_m} = \frac{1}{\tau}\sum_{t=0}^{\tau-1} g_m(t), \forall m \in \{1,2,\ldots,M\}$$

$$\bar{z} = \frac{1}{\tau}\sum_{t=0}^{\tau-1} z(t)$$

$\min \bar{z}$

- $\overline{g_m} \le 0, \forall m \in \{1,2,\ldots,M\}$
- other constraints within current time slot
- queue stability

$$L(Q(t+1)) - L(Q(t)) = \frac{1}{2}\sum_{i=1}^{N}\left[Q_i^2(t+1) - Q_i^2(t)\right] = \frac{1}{2}\sum_{i=1}^{N}\left[\left(\max[Q_i(t) - b_i(t)] + a_i(t)\right)^2 - Q_i^2(t)\right]$$

$$\le \frac{1}{2}\sum_{i=1}^{N}\frac{a_i^2(t)+b_i^2(t)}{2} + \sum_{i=1}^{N} Q_i(t)[a_i(t) - b_i(t)]$$

where we use the fact that

$$\left(\max[Q_i(t) - b_i(t)] + a_i(t)\right)^2 \le Q_i^2(t) + a_i^2(t) + b_i^2(t) + 2Q_i(t)[a_i(t) - b_i(t)]$$

Finally, we have

$$\triangle(Q(t)) = \mathrm{E}\left(L(Q(t+1)) - L(Q(t)) | Q(t)\right) \le B + \sum_{i=1}^{N} Q_i(t)\mathrm{E}[a_i(t) - b_i(t) | Q_i(t)]$$

The Lyapunov Optimization and Lyapunov Drift

❖ Details

- **Lyapunov Function:** $L\big(Q(t)\big) = \frac{1}{2}\sum_{i=1}^{N} Q_i^2(t)$
- **Lyapunov Drift:**
  - i.i.d.: One-time slot $\triangle\big(Q(t)\big) = \mathrm{E}\Big(L\big(Q(t+1)\big) - L\big(Q(t)\big)|Q(t)\Big)$
  - Non i.i.d.: *T*-time slot $\triangle\big(Q(t)\big) = \mathrm{E}\Big(L\big(Q(t+T)\big) - L\big(Q(t)\big)|Q(t)\Big)$

➢ **Tackle the time average constraint $\overline{g_m} \le 0, \forall m \in \{1,2,\dots,M\}$: introducing the virtual queues**

$$Z_m(t+1) = max\{Z_m(t), 0\} + g_m(t), \forall m \in \{1,2,\dots M\}$$

* "Virtual" means $Z_m(t)$ does not represent a real data backlog
* A way to turn the time average inequality constraint into a pure queue stability problem

- **Lyapunov Function:** $L\big(\Theta(t)\big) = \frac{1}{2}\sum_{i=1}^{N} Q_i^2(t) + \frac{1}{2}\sum_{m=1}^{M} Z_m^2(t)$
- **Lyapunov Drift:**
  - i.i.d.: One-time slot $\triangle\big(\Theta(t)\big) = \mathrm{E}\Big(L\big(\Theta(t+1)\big) - L\big(\Theta(t)\big)|\Theta(t)\Big)$
  - Non i.i.d.: *T*-time slot $\triangle\big(\Theta(t)\big) = \mathrm{E}\Big(L\big(\Theta(t+T)\big) - L\big(\Theta(t)\big)|\Theta(t)\Big)$

The Lyapunov Optimization and Lyapunov Drift

- **Drift plus penalty (minus reward) function:**

$$\triangle\left(Q(t)\right) + \mathrm{VE}\left(z(t)|Q(t)\right)$$

V: 1. non-negative control parameter
2. larger V leads to more emphasis on penalty minimization

$$\triangle\left(Q(t)\right) + \mathrm{VE}\left(z(t)|Q(t)\right) \le B + \mathrm{VE}\left(z(t)|Q(t)\right) + \sum_{i=1}^{N} Q_i(t)\mathrm{E}\left(a_i(t) - b_i(t)|Q(t)\right)$$

Obtain the control decision by minimizing the right-hand-side of the inequality at every time slot t

The Lyapunov Optimization and Lyapunov Drift

- **The time-average queue length yields:**

i.i.d.

$$\lim_{\tau\to\infty} \sup \frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{i=1}^{N} \mathrm{E}\{Q_i(t)\} \le \frac{B + V[z^{opt} - z_{min}]}{\varepsilon}$$

Non i.i.d.

$$\lim_{\tau\to\infty} \sup \frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{i=1}^{N} \mathrm{E}\{Q_i(t)\} \le \frac{DT}{\varepsilon} + \frac{V[z^{opt} - z_{min}]}{\varepsilon} + \frac{T-1}{2}\sum_{i=1}^{N} \max[a_i^{max}, b_i^{max}]$$

- **Minimum penalty bound:**

i.i.d.

$$\lim_{\tau\to\infty} \sup \frac{1}{\tau}\sum_{t=0}^{\tau-1} \mathrm{E}\{z(t)\} \le z^{opt} + \frac{B}{V}$$

Non i.i.d.

$$\lim_{\tau\to\infty} \sup \frac{1}{\tau}\sum_{t=0}^{\tau-1} \mathrm{E}\{z(t)\} \le z^{opt} + \frac{D}{V}$$

[O(V),O(1/V)] tradeoff

- Tuning V to be sufficiently large so that the minimum penalty can be achieved (+)
- Time average queue backlog bound increase linearly with V. (-)

THANK YOU!

We are impatient to work with you. We always welcome applications from visiting scholars at all levels (students, faculty, postdocs) who are interested to spend some time in our lab and get involved in our ongoing research activities.