Martingale Theory

Romano Fantacci,
and Benedetta Picano

Martingale Process

A discrete-time martingale is a discrete-time stochastic process (a sequence of random variables) $X_1, X_2, X_3 \cdots$ that satisfies for any time n

$$E[|X_n|] < \infty$$

$$E(X_{n+1} | X_1 \cdots X_n) = X_n$$

Due to the linearity of expectation,

$$E\left(X_{n+1} - X_n \middle| X_1, \cdots, X_n\right) = 0$$

or

$$E\left(X_{n+1} \middle| X_1, \cdots, X_n\right) - X_n = 0$$

Martingale Process

A continuous-time martingale satisfies

$$E\left(|Y_t|\right) < \infty,$$

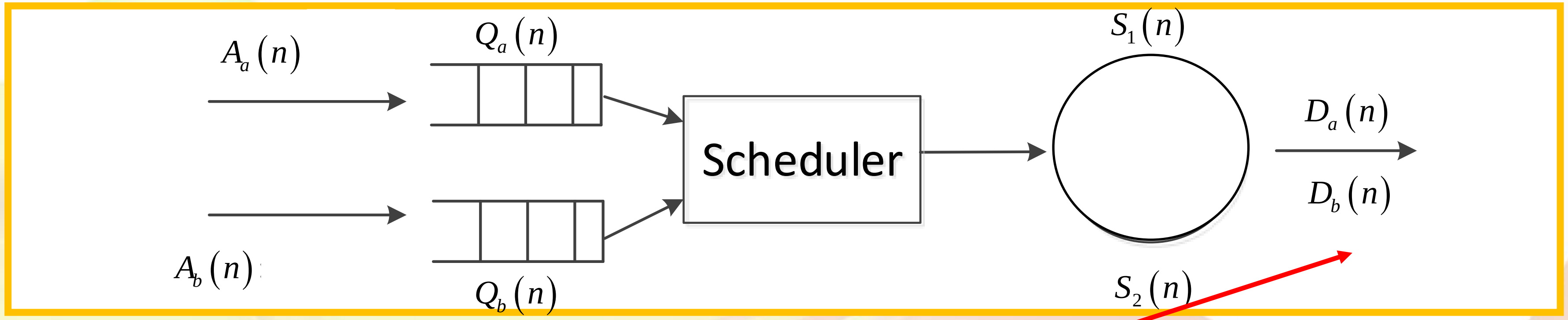
$$E\left(Y_t \,\middle|\, \{X_\tau, \tau \le s\}\right) = Y_s, \forall s \le t.$$

The conditional expectation of an observation at time t, given all the observations up to time s , is equal to the observation at time s.

Discrete time: $E\left(X_{n+1} \,\middle|\, X_1, \quad , X_n\right) \le X_n$

Continuous time: $E\left(X_t \,\middle|\, \{X_\tau : \tau \le s\}\right) \le X_s, \forall s \le t$

Martingale Process



What is the end-to-end delay bound of two different data flows when traversing the servers under a specific scheduling policy such as first in first out (FIFO) or earliest deadline first (EDF)?

Main Actors

Bivariate arrival process

$$A(m,n) = A(n) - A(m) = \sum_{k=m+1}^{n} a_k$$

Bivariate service process

$$S(m,n) = S(n) - S(m) = \sum_{k=m+1}^{n} s_k$$

Departure process

$$D(n) \geq \inf_{0 \leq k \leq n} \{A(k) + S(k,n)\} = A \otimes S(n)$$

Main Actors

Backlog process : $Q(n) = A(n) - D(n) = \sup_{0 \le k \le n} \{A(k,n) - S(k,n)\}$

Delay process : $W(n) = \inf \{k \ge 0 : A(n-k) \le D(n)\}$

$$\inf \left\{k \ge 0 : \sup_{0 \le k \le n} \{A(k,n) - S(n)\} \le 0\right\}$$

Main Actors

(Supermartingale Envelope for Arrivals). The arrival flow $A(n)$ admits a supermartingale envelope, if the process

$$h_a(a_n)e^{\theta(A(n)-nK_a)} \le M_a(n), \text{ for } A(n),$$

is bounded by the arrival supermartingale $M_a(n)$

(Supermartingale Envelope for Services). The service process admits a supermartingale envelope, if the process

$$h_s(s_n)e^{\theta(S(n)-nK_s)} \le M_s(n), \text{for } S(n),$$

is bounded by the service supermartingale $M_s(n)$

Per-Flow Delay for FIFO

The FIFO server schedules the arrival data of $A_a(n)$ and $A_b(n)$ accordingly to their arrival times. The extended bivariate stochastic service process for FIFO scheduling algorithm can be presented as

$$\begin{cases} S_a(m,n) = \left[S(m,n) - A_b(m,n-x) \right]_+ 1_{\{n-m>x\}} \\ S_b(m,n) = \left[S(m,n) - A_a(m,n-x) \right]_+ 1_{\{n-m>x\}} \end{cases}$$

Per-Flow Delay for FIFO

Arrival processes bounded by $M_{A_a}(n), M_{A_b}(n)$

Service process bounded by $M_S(n)$

$$M_{A_a}(n) = h_{A_a}(a_n) e^{\theta^*\left(A_a(k,n)-(n-k)K_{A_a}\right)}, \text{for } n \geq k$$

$$M_{A_b}(n) = h_{A_b}(a_n) e^{\theta^*\left(A_b(n)-nK_{A_b}\right)}, \text{for } n \geq 0$$

$$M_S(n) = h_S(s_n) e^{\theta^*(nK_S - S(n))}, \text{for } n \geq 0$$

By stationary, time-shifted supermartingale process is also a supermartingale

In accordance with the independence assumption, for $\{k, k+1, \cdots\}$

$$M_k(n) = h_{A_a}(a_n) e^{\theta^*\left(A_a(k,n)-(n-k)K_{A_a}\right)} \cdot h_{A_b}(a_n) e^{\theta^*\left(A_b(n)-nK_{A_b}\right)} \cdot h_S(s_n) e^{\theta^*(nK_S - S(n))}$$

is a supermartingale

Per-Flow Delay for FIFO

Backlog process:

$$B_a(n) \le \sup_{0\le k\le n}\left\{A_a(k,n)-(n-k)K_{A_a}+A_b(n)-nK_{A_b}+nK_S-S(n)\right\}$$

$$P\left(B_a(n)\ge\sigma\right)=P(N<\infty)=P\left(N<k\,\middle|\,k\to\infty\right)$$

$$\le\frac{E\left[M_k(k)\right]}{H}e^{-\theta^*\sigma}\le\frac{E\left[M_{A_a}(0)\right]E\left[M_{A_b}(0)\right]E\left[M_S(0)\right]}{H}e^{-\theta^*\sigma}$$

Delay process distribution:

$$P\left(W_a(n)\ge k\right)\le P\left(\sup_{0\le k\le n}\left\{A_a(k,n)-S_a(n)\right\}\ge 0\right)$$

$$\le\frac{E\left[M_{A_a}(0)\right]E\left[M_{A_b}(0)\right]E\left[M_S(0)\right]}{H}e^{-\theta^* kK_S}$$

Per-Flow Delay for EDF

An EDF server schedules by combing the relative deadlines and the arriving process. In EDF scheduling, all data are transmitted in the order of their remaining deadlines.

$$\begin{cases} S_a(m,n) = \left[S(m,n) - A_b\left(m, n - x + \min\{x, y\}\right)\right]_+ 1_{\{n-m>x\}} \\ S_b(m,n) = \left[S(m,n) - A_a\left(m, n - x + \min\{x, y\}\right)\right]_+ 1_{\{n-m>x\}} \end{cases}$$

where y denotes the difference of two arrivals' deadlines.

Per-Flow Delay for EDF

Delay process distribution for A_a (n):

for y>0,

$$P\left(W_a(n) \geq k\right) \leq \frac{E\left[M_{A_a}(0)\right]E\left[M_{A_b}(0)\right]E\left[M_S(0)\right]}{H} e^{-\theta^*\left(kK_S - \min\{k,y\}\right)}$$

for y<0,

$$P\left(W_a(n) \geq k\right) \leq \frac{E\left[M_{A_a}(0)\right]E\left[M_{A_b}(0)\right]E\left[M_S(0)\right]}{H} e^{-\theta^*\left(kK_S - yK_b\right)} + \frac{E\left[M_{A_a}(0)\right]E\left[M_S(0)\right]}{H} e^{-\theta^* kK_S}$$

THANK YOU !

We are impatient to work with you. We always welcome applications from visiting scholars at all levels (students, faculty, postdocs) who are interested to spend some time in our lab and get involved in our ongoing research activities.