Elements of Stochastic Network Calculus

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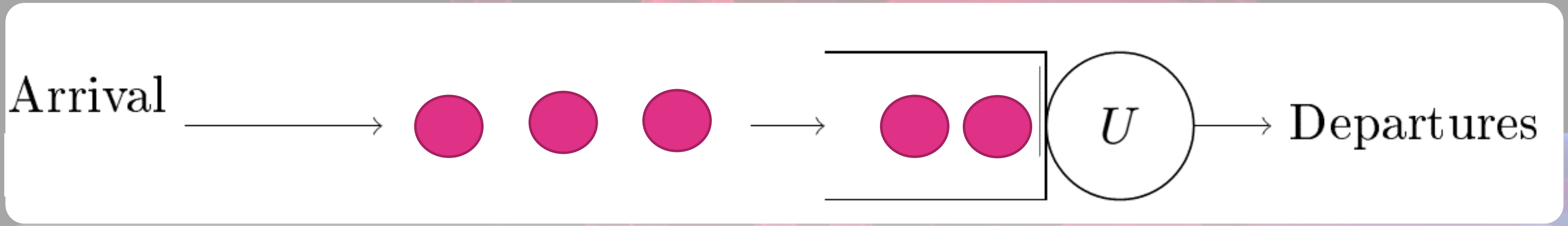
Stochastic Network Calculus

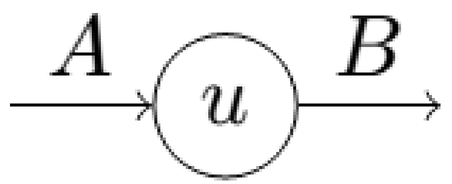


Queueing theory provides useful analytical model to plan and analyze the system performance.

The stochastic network calculus (SNC) offers stochastic service guarantee, i.e., a service is guaranteed with a probability.

Some requests may violate the constraint

Min-Plus Algebra



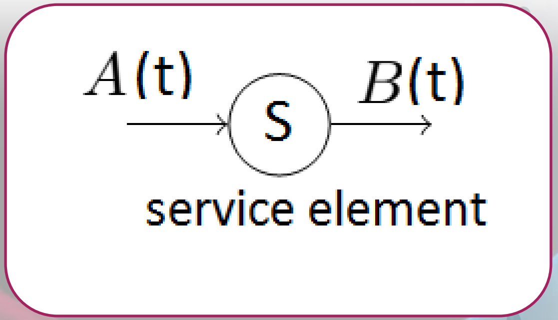
$$A \otimes B(t) \mapsto \inf_{0 \le s \le t} \{A(s) + B(t-s)\}$$ *Convolution operator*

$$A \oplus B(t) \mapsto A(t) + B(t)$$ *Multiplexing operator*

$$A \oslash B(t) \mapsto \sup_{0 \le s} \{A(t+s) - B(s)\}$$ *Deconvolution operator*

$$A \ominus B(t) \mapsto A(t) - B(t)$$ *Subtraction operator*

Arrivals Envelope



- Let A be the cumulative arrivals flow. For any times $0 \le s \le t$,

$$A(t) - A(s) \le \rho(t - s) + b$$

- The deterministic envelope cannot take advantage of the statistical nature of traffic

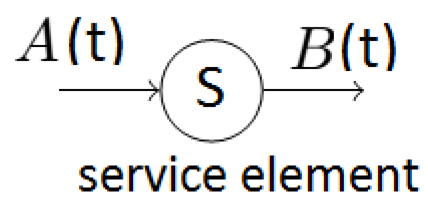
- **Solution:** introducing the Moment Generating Function (MGF)

$$M_A(\theta, t - \tau) = E\left[e^{\theta A(\tau,t)}\right]$$

- MGF envelope

$$E\left[e^{\theta A(\tau,t)}\right] \le e^{\theta(\rho(t-\tau)+\sigma)}$$

Service Envelope



- A service element has the input-output pair *A* and *B*.
- The service element provides a **service curve**

$$B(t) - A(s) \geq U(t-s)$$

$$B(t) \geq A \otimes U(t).$$

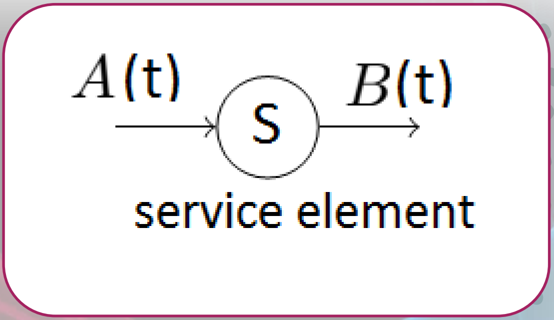
- A deterministic definition of service envelope for all t ≥τ ≥ 0 is

$$S(\tau, t) \geq \rho(t - \tau) - b$$

- An envelope of the MGF can be defined for θ ≥ 0 as

$$E\left[e^{-\theta S(\tau,t)}\right] \leq e^{-\theta(\rho(t-\tau)-\sigma)}$$

Service Envelope



- Statistical service envelopes can be represented by the Exponentially Bounded Fluctuation (EBF) model with parameters ρ > 0, b ≥ 0 as

$$P[S(\tau,t) > \rho(t-\tau) - b] \leq \varepsilon(b)$$

$$\varepsilon(b) = \alpha e^{-\theta b} \triangleq e^{\theta\sigma} e^{-\theta b}$$

- By exploiting the following law

$$P[X \leq x] \leq e^{\theta x} E[e^{-\theta X}]$$

- We have

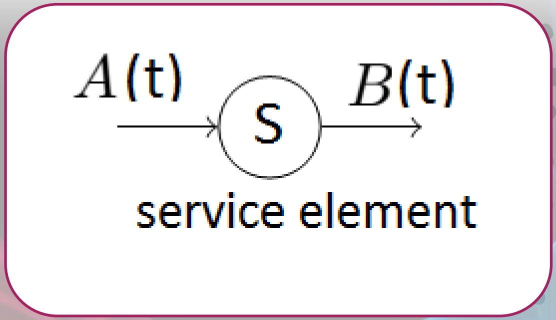
$$\varepsilon(b) = e^{\theta\sigma} e^{-\theta b}$$

$$P[S(\tau,t) > \rho(t-\tau) - b] \leq \varepsilon(b)$$

⇩

$$P[\exists\tau \in [0,t]: S(\tau,t) > \rho'(t-\tau) - b] \leq \varepsilon'(b)$$

Service Envelope



- By the union bound

$$P[\exists i: X_i \geq x] \leq \sum_i P[X_i \geq x]$$

- We have

$$P[\exists \tau \in [0,t]: S(\tau,t) > \rho'(t-\tau) - b]$$

Let $\rho' = \rho - \delta$.

$$P[S(\tau,t) > \rho(t-\tau) - b] \leq \varepsilon(b)$$

$$\varepsilon(b) = \alpha e^{-\theta b} \triangleq e^{\theta\sigma} e^{-\theta b}$$

$$\leq \sum_{\tau=0}^{t} e^{\theta\sigma} e^{-\theta(b+\delta(t-\tau))}$$

$$\leq e^{\theta\sigma} e^{-\theta b} \sum_{\tau=0}^{\infty} e^{-\theta\delta\tau}$$

$$= \frac{e^{\theta\sigma} e^{-\theta b}}{1 - e^{-\theta\delta}} = \varepsilon'(b)$$

where $\vartheta > 0$ and $\delta > 0$ are free parameters that can be optimized.

- We obtain the relationship between the θ, σ, ε′, δ and *b*.

Backlog and Delay Bound

- Consider arrivals with envelope $A(\tau,t) \le \rho'_A(t-\tau) + b_A$
- Consider service with envelope $S(\tau,t) \ge \rho'_S(t-\tau) + b_S$
- Backlog bound $B(t) \le A(t) \oslash S(t) = \max_{\tau\in[0,t]}\{A(\tau,t) - S(\tau,t)\}.$

$$b = \max_{\tau\in[0,t]}\{\rho'_A(t-\tau) + b_A - [\rho'_S(t-\tau) - b_S]_+\}.$$

$$P[B(t) > b] \le \varepsilon'_A(b_A) + \varepsilon'_S(b_S) = \varepsilon'$$

$$b = b_A + b_S \frac{\rho_A + \delta}{\rho_S - \delta}$$

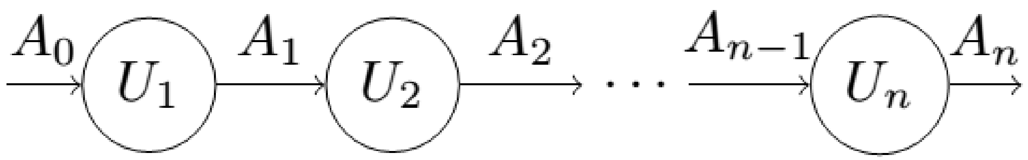
- Delay bound $W(t) \le \min\left\{\omega \ge 0: \max_{\tau\in[0,t]}\{A(\tau,t) - S(\tau,t+\omega)\} \le 0\right\}$

$$\omega = \frac{b_A + b_S}{\rho_S - \delta}$$

$$b_A = \sigma_A - \frac{1}{\theta}\left(\ln\left(\frac{\varepsilon'}{2}\right) + \ln\left(1 - e^{-\theta\delta}\right)\right)$$

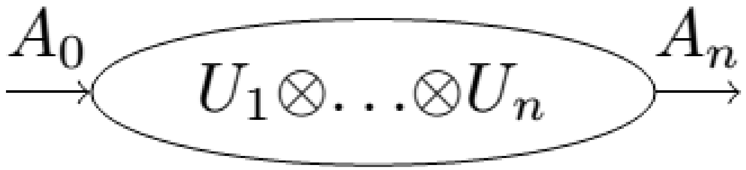
$$b_S = n\sigma_S - \frac{1}{\theta}\left(\ln\left(\frac{\varepsilon'}{2}\right) + n\ln\left(1 - e^{-\theta\delta}\right)\right)$$

Tandem Systems



- The whole system has the service curve

$$U_1 \otimes \ldots \otimes U_n .$$



$$A_n \geq A_0 \otimes U_1 \otimes \ldots \otimes U_n .$$

Tandem Systems

The MGF of the min-plus convolution of two statistically independent and stationary service processes is given by

$$E\left[e^{-\theta(S_1 \otimes S_2)(\tau,t)}\right] = E\left[e^{-\theta \min_{v\in[\tau,t]}\{S_1(\tau,v)+S_2(v,t)\}}\right]$$

$$\leq \sum_{v=\tau}^{t} E\left[e^{-\theta S_1(\tau,v)}\right] E\left[e^{-\theta S_2(v,t)}\right]$$

$$= \sum_{v=0}^{t-\tau} M_{S_1}(-\theta, v) M_{S_2}(-\theta, t-\tau-v)$$

$$= M_{S_1} * M_{S_2}(-\theta, t-\tau)$$

Tandem Systems

The MGF of the service process of an n nodes network follows the rule

$$M_{S_{net}}(-\theta,t) \leq M_{S_1} * M_{S_2} * \ldots * M_{S_n}(-\theta,t)$$

$$= \sum_{\tau_i \geq 0: \sum_{i=1}^{n} \tau_i = t} M_{S_1}(-\theta,\tau_1) M_{S_2}(-\theta,\tau_2) \ldots M_{S_n}(-\theta,\tau_n)$$

THANK YOU!

We are impatient to work with you. We always welcome applications from visiting scholars at all levels (students, faculty, postdocs) who are interested to spend some time in our lab and get involved in our ongoing research activities.